**Fold Problem.** A sheet of paper has a height H and width W. One corner is folded so that the tip of the fold just touches the opposite side. Show that the minimum length of the fold equals $\frac{3\sqrt{3}\,W}{4}$.

The figure suggests one way to approach this problem. The length of the fold (z) is the hypotenuse of the right triangle 123. The Pythagorean relation for the triangle is

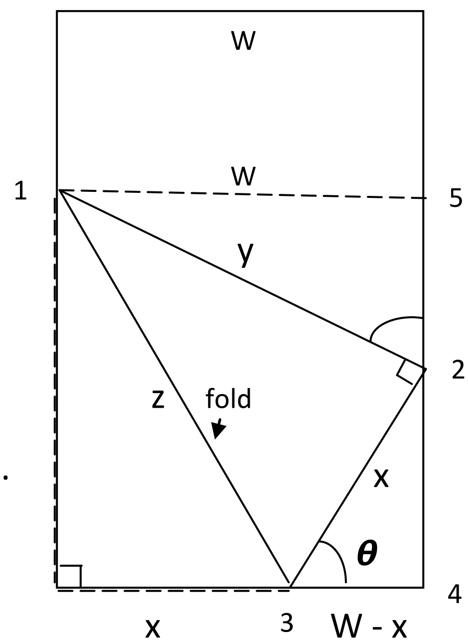
$$z^2 = x^2 + y^2$$

The plan is to figure out relations that let us express x and y in terms of the angle $\boldsymbol{\theta}$.

Then we can use a derivative to find the value of $\boldsymbol{\theta}$ that minimizes z.

Two other right triangles are key to the solution.

**The triangles 243 and 152 are similar**.




To prove this show that the angle $\boldsymbol{\theta}$ (234) is equal to the angle 125. It follows that

$$\frac{W}{y} = \sin\boldsymbol{\theta} \qquad \frac{W-x}{x} = \cos\boldsymbol{\theta} \;\;=>\;\; \frac{W}{x} = 1 + \cos\boldsymbol{\theta}$$

These two equations let you express x and y in terms of $\boldsymbol{\theta}$. This in turn lets you express $z^2$ in terms of $\boldsymbol{\theta}$. What comes next? Set $(\frac{dz}{d\theta}) = 0$ ? That would work, but it leads to ugly equations. If z is a minimum $1/z^2$ is a maximum so setting the derivative of $1/z^2$ equal to zero finds the maximum of $1/z^2$ and thereby the minimum of z. This leads to a quadratic equation for $\cos\boldsymbol{\theta}$ that gives $\cos\boldsymbol{\theta} = \frac{1}{3}$. This makes $\sin\boldsymbol{\theta} = \frac{\sqrt{8}}{3}$ and $z = \frac{3\sqrt{3}\,W}{4}$. Done? Rework it. Express $1/z^2$ in terms of $t = \cos\boldsymbol{\theta}$. Differentiate $1/z^2$ with respect to t and solve for t. You get a quadratic with the solution t = 1/3. There's always a smoother way to solve a problem.

To prove that remark, here is yet another solution: The triangles 123 and 163 are congruent because they are opposite sides of the fold. Use this fact to show that the angles 213 and 316 equal $\boldsymbol{\theta}/2$. This lets you write the equation

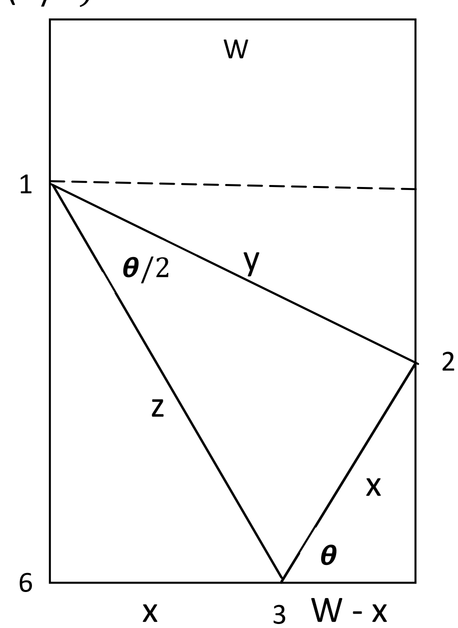
$$\frac{x}{z} = \sin(\boldsymbol{\theta}/2)$$

Multiply this by a previous equation $\frac{W}{x} = (1 + \cos\boldsymbol{\theta})$. This eliminates x and gives an equation relating $\boldsymbol{\theta}$ and z

$$\frac{W}{x}\frac{x}{z} = (1 + \cos\boldsymbol{\theta})\sin(\boldsymbol{\theta}/2) \quad => \quad \frac{W}{z} = (1 + \cos\boldsymbol{\theta})\sin(\boldsymbol{\theta}/2)$$

Use the double angle formula to replace $(1 + \cos\boldsymbol{\theta})$ by $2[\cos(\boldsymbol{\theta}/2)]^2$

$$\frac{W}{z} = 2[\cos(\boldsymbol{\theta}/2)]^2 \sin(\boldsymbol{\theta}/2) = 2(1 - [\sin(\boldsymbol{\theta}/2)]^2)\sin(\boldsymbol{\theta}/2)$$




Setting $\sin(\boldsymbol{\theta}/2) = \text{s}$ converts this to

$$\frac{W}{z} = 2s(1 - s^2) = 2(s - s^3)$$

Setting the derivative of $\frac{W}{z}$ equal to zero locates the maximum of $\frac{W}{z}$ and the minimum of z. This gives

$$1 - 3s^2 = 0 \quad => \quad s^2 = 1/3$$

Plugging this back into

$$\frac{W}{z} = 2s(1 - s^2)$$

gives the minimum value, $z = \frac{3\sqrt{3}\,W}{4} = 1.299\ W$

I discovered this problem online in a group posted by Professor Duane Kouba kouba@math.ucdavis.edu. It is different than the typical max-min problem. Notice that the solution does not depend on the height of the paper (H). This becomes more apparent **after** you have solved the problem. Attempting to find a solution using the height H introduces a red herring – and leads you astray.

All done? Not quite. There is still an unanswered question. Why should there be a minimum? It's not hard to see that the fold gets longer as $\boldsymbol{\theta}$ is made smaller. It's easy to use a sheet of paper to convince yourself that the fold gets longer and longer as $\boldsymbol{\theta}$ approaches zero. The other limiting value of $\boldsymbol{\theta}$ is 90 degrees and this leads to z = $\sqrt{2}$ W = 1.414 W which is not much greater than the minimum value 1.299 W. So….. the question is this:

Is there an **intuitive** reason that explains why there is a minimum?

Your insights will be welcomed. don@donckelly.com

The current answer to the question above is "sort of", but not a satisfying one.

To complete the record, here is a third solution, the first one I cobbled together.

$$\frac{W}{y} = sin\boldsymbol{\theta} \qquad \frac{W-x}{x} = \cos\boldsymbol{\theta} \quad => \quad \frac{W}{x} = 1 + \cos\boldsymbol{\theta}$$

$$z^2 = x^2 + y^2 = x^2\{1 + (y/x)^2\} = x^2\{1+ \frac{(1+cos\theta)^2}{[sin\theta]^2}\} = x^2\{\frac{\mathbf{2(1+cos\theta)}}{[sin\theta]^2}\} = x^2\{\frac{\mathbf{2}}{(1-\cos\theta)}\}$$

$$z^2 = \frac{\mathbf{2}x^2}{(2-\frac{W}{x})}$$

If z is a minimum so also is $z^2$. Setting the derivative of $z^2$ to zero gives

$$\frac{4x-3W}{(2-\frac{W}{x})^2} = 0 \qquad => \qquad x = \frac{3W}{4}$$

Plug in $x = \frac{3W}{4}$ to get the minimum value of $z^2$

$$z^2 = \frac{27}{16}W^2 \quad => \quad z = \frac{3\sqrt{3}\,W}{4}$$

From $\frac{W}{x} = 1 + \cos\boldsymbol{\theta}$ it follows that $\cos\boldsymbol{\theta} = 1/3$ and $\boldsymbol{\theta} \approx 70.5$ degrees.